J. Fluid Mech. (1996), vol. 328, pp. 49–66

Self-similar multi-layer exchange flow through a contraction

By ANDERS ENGQVIST

Department of Systems Ecology, Stockholm University, S-106 91 Stockholm, Sweden
e-mail: ae@system.ecology.su.se

(Received 25 May 1994 and in revised form 2 July 1996)

The multi-layer exchange equations for gravitationally driven flows between two basins with stable Boussinesq type of stratification in discrete layers, specified far upstream on either side of a connecting strait, result in a hydraulic control condition that must be satisfied at the narrowest part of the contraction, the control point. If one stagnant layer is present at the control point, the control condition that applies to all layers collectively may be separated into two such conditions that apply independently to two groups of layers going in opposite directions separated by the stagnant layer. Such bidirectional flow regimes exist if the structure of the prespecified density profiles permits each of the opposing groups to vertically reduce their thickness by the ratio 2/3 relative to their upstream thicknesses, leaving space for the stagnant layer to protrude through the contraction. Under these restrictions, the bidirectional flow is controlled by the fastest propagating wave mode and the stationary solution then relies on the superposition of two previously known unidirectional self-similar flow regimes that are completely decoupled. Techniques for their numerical computation are presented. The transition into loosely coupled and fully coupled flow is discussed. The decoupling principle also applies when several non-adjacent stagnant layers are simultaneously present at control in which case multiple groups of decoupled layers flow in alternating directions.

1. Introduction

Beginning with Stommel & Farmer (1952, 1953) continued studies bear witness to the importance and intricacies of the two-layer exchange problem. One reason is that in many practical instances, for example ventilation of estuaries, two-layer approximation serves as an adequate idealization. Typical cases could be small-scale coastal inlets that are subjected to over-mixing so that densimetrically homogeneous water bodies are formed on either side of a connecting strait. In places where such idealized homogeneous conditions do not prevail, the two-layer approximation may be poor and multiple layers with discrete densities provide a more realistic description. As the number of strata increases, continuous stratification is the limiting result. This enhanced realism to describe the stratification is, however, not accompanied by available methods to compute the resulting flow. For the few cases with known solutions, mainly with prescribed flow in one section, the mathematical treatment is naturally quite different compared to the discrete case. Early contributions were presented by Craya (1951) followed by Long (1953, 1954, 1955). Killworth (1992) provided a procedure to facilitate the conversion of the continuous equations to the corresponding discrete multi-layer formulation. He also treated bidirectional flows for continuously stratified profiles and showed that when the external (barotropic) mode is eliminated, the interface between the opposing flows will be flat. This situation does not occur in natural strait exchange flows where the external mode plays an integral part in the flow adjustment (e.g. Stigebrandt 1990).

The first work on multi-layered flows which presented the control criterion in the form of a determinant equation (Benton 1954) applies to bidirectional flows. Over the following decades the focus has mainly been on unidirectional multi-layered flow with prespecified velocities in an upstream section (e.g. Su 1976; Lee & Su 1977; Baines 1988). These known upstream velocities provide a solid starting point from which computational schemes may be integrated. Since these schemes rely on iterative division with the upstream velocities, this advantage is lost if the boundary condition involves fluid at rest, as is the case when fluid is withdrawn form a large reservoir.

Multi-layered selective withdrawal was first treated by Wood (1968). Starting with two flowing layers surrounded by stagnant layers, he found a similarity solution with a height reduction factor (i.e. the ratio between the layer thickness at the contraction and the specified thickness of the same layer far upstream) of 2/3 and one control located in the narrowest section. This could be generalized to cover a continuously and stably stratified flow. Binnie (1972) used Hugoniot's method to produce the same results for two-layer flow but with much simplified algebra. Benjamin (1981) verified this result by using the same momentum transfer function as Benton (1954), and proved that the momentum transfer per unit span (flow force) integrated over the active layers is maximal for flow subjected to control. A review of laboratory studies has been provided by McClimans (1990).

The present paper treats the case where there is no sill and the bottom of the connecting channel is horizontal. The problem considered is mainly the same as studied by Stigebrandt (1990), which was suggested by the often occurring practical situation when the ventilation of coastal embayments comes into focus: a sea with slowly varying density surfaces including the interface to the atmosphere is connected to an embayment by a constricting strait.

The presentation is confined to achieving a theoretically sound method to compute quasi-stationary exchange through a connecting strait with varying width, located between two sufficiently large bodies of stably stratified water. The exact solutions will be limited to the cases when one (or more non-adjacent) stagnant layer(s) is (are) present at the maximal contraction. The proposed method is based on self-similar solutions. Even though it seems potentially capable of handling a sill that coincides with the maximal contraction, this represents a more complicated case in comparison to a flat bottom (Farmer & Armi 1986). Therefore the presentation will focus on the latter case. The flow at the strait should comply with the control criterion and should be matched through hydraulic jumps with the prespecified downstream stratification. A sketch is provided in figure 1.

An outline of this paper is that first (§2) the basic shallow water equations (i.e. the pressure field is hydrostatic everywhere, jumps possibly excepted) are reviewed and from these the control criterion is derived. The derivation up to equation (12*a*, *b*) is similar to that of Benton (1954). The various forms the control condition takes when a stagnant layer is present at the control point are given in §3. A decoupling property of the control criterion is then examined, on which the formulation of the stagnation conditions for self-similar decoupled flows in §4 is based. The results are discussed in §5 followed by a summary.

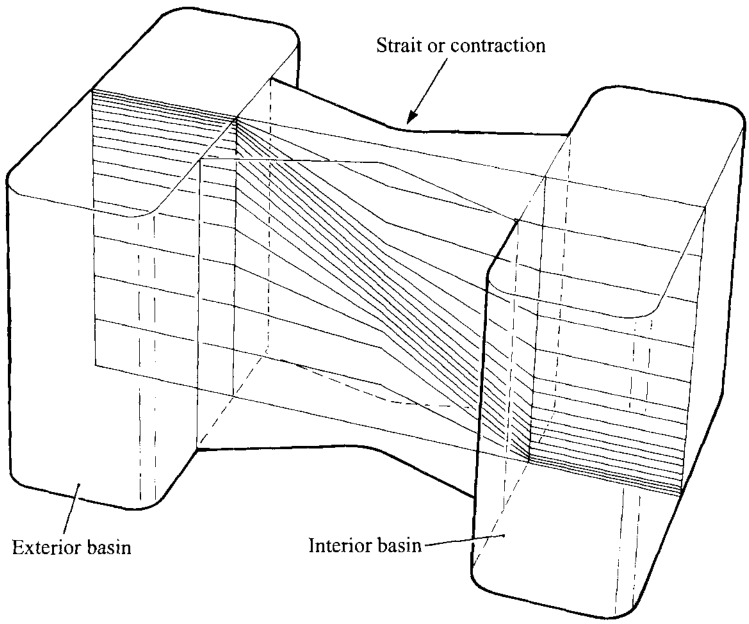

FIGURE 1. Sketch of the problem formulation. The different stratification in the reservoirs has been indicated by connecting the isopychs through the contraction.

2. Problem formulation and basic equations

The exchange of water through a contraction (with given geometric properties) is a function of the density fields on either side (for example in the interior of the embayment and in the external open sea) and the sea level difference. It is subsequently assumed that the variation in time of the external baroclinic and barotropic forcing is so slow that the flow could be considered as quasi-stationary. This means that the basins must be sufficiently large that the response time of the embayment to water exchange is slow in comparison to travel time of the water parcels exchanged (Wood 1970). For the specified density distribution one would want to compute the resulting flow for as wide a range of the barotropic component as possible. This part of the flow is a function of sea level difference between the basins, and leaves two alternative but equally valid perspectives. Either one regards the barotropic flow as specified so that the sea level difference adjusts to produce this flow, or one regards the sea level difference as known and determining the barotropic flow. Keeping in mind the assumption of quasi-stationary timescales, the latter perspective will be chosen here. The sea level difference between the basins will be denoted da (figure 2) in accordance with Stigebrandt's (1990) nomenclature, and it will be retained as a parameter free to vary within the constraint that the external Froude number is considerably less than unity so that a rigid-lid approximation applies. Hydraulic jumps and/or separations may occur downstream of the maximum contraction. The stratification is assumed stable on either side of the strait and is represented by n layers spanning the same density range from ρ_1(top) to ρ_n (bottom). It is tacitly assumed Boussinesqian in the sense that

$$0 < \rho_n - \rho_1 \ll \tfrac{1}{2}(\rho_n + \rho_1). \tag{1}$$

It is further assumed here that the topography of the connecting channel is sufficiently long and smooth that the hydrostatic assumption (i.e. the shallow water equation) is

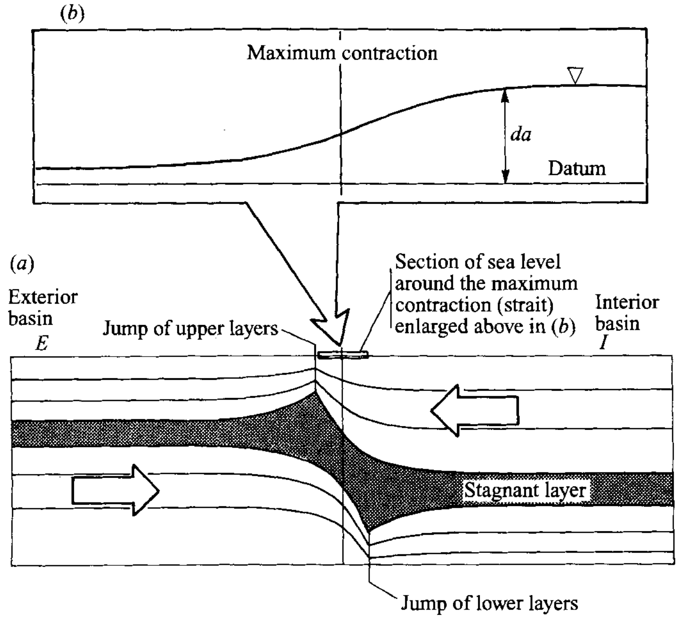

FIGURE 2. (*a*) A sketch of the streamlines along isopychs of a seven-layer flow regime with the fourth layer stagnant. The jumps are indicated. (*b*) The infinitesimal sea level elevation (*da*) around the narrowest part of the contraction is shown magnified in the vertical direction.

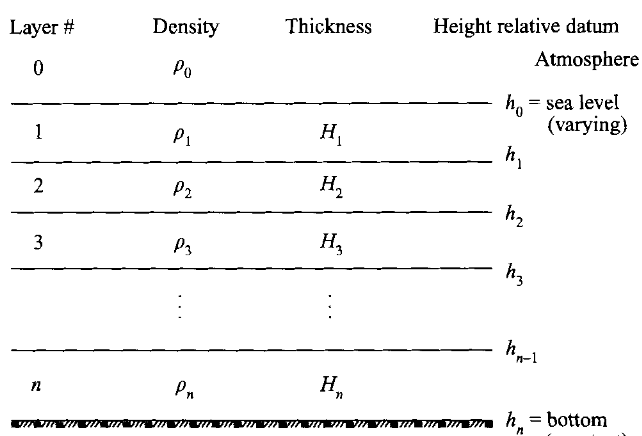

FIGURE 3. Geometric convention and nomenclature.

valid. Cross-channel velocities are assumed uniform (no occurrence of *vena contracta*) and lateral accelerations negligible. The walls are assumed vertical but this assumption will be relaxed later. Viscous and other frictional effects are neglected entirely. This may be unrealistic in long narrow contractions, since friction acts to reduce the available area by formation of lateral boundary layers (Wood 1970; Armi & Williams 1993). Rotational effects are not discussed. The interested reader is referred to Gill (1977) for single-layer flow and Dalziel (1991) for two-layer flow. The density ρ_0, denoting the density of a possible atmosphere on top of the surface layer, is set to zero.

With the notation in figure 3 the momentum equation for the *i*th of a total of $n+1$ layers (the zeroth layer denotes the atmosphere) may be written using the comma notation for the partial derivatives with regard to the horizontal (x) and time (t) coordinates if a hydrostatic pressure field is assumed:

$$\rho_i(U_{i,t}+U_i U_{i,x}) = -g \sum_{j=1}^{i} \Delta\rho_j h_{j-1,x}; \tag{2}$$

U_i and ρ_i denote the homogeneous velocity and the density of the ith layer respectively. Further the $\Delta\rho_j = (\rho_j-\rho_{j-1})$ are positive quantities, so that the stratification is stable. The height relative to an arbitrary datum of the lower interface of the ith layer is denoted h_i (positive upwards), thereby denoting the free surface as h_0 and the bottom contour h_n. The latter is assumed constant. The small sea level deviation from the horizontal will, in all practical cases, be small in comparison to the height of the total water column, and thus the sum of the thicknesses of all layers may be regarded as constant in the horizontal. The sea surface is thus regarded as geometrically flat (i.e. its deviation only negligibly affects the vertical position of the other layers) but it still permits a significant barotropic mode by affecting the pressure profiles. This is essentially a rigid-lid assumption. Volume conservation of an incompressible fluid requires that

$$W_i H_{i,t} = (U_i W_i H_i)_{,x}, \tag{3}$$

where W_i denotes the width (assumed known) and H_i the thickness of layer i. The geometry (figure 3) gives for the layer thicknesses

$$H_i = h_{i-1}-h_i. \tag{4}$$

Thus there are two equations for two unknowns (U_i and H_i or h_{i-1}) for each layer and together with boundary conditions (i.e. the known densities and layer thickness of the two large reservoirs connected by a channel) the solution should be possible to obtain by integration. However, the inherent nonlinearities are such that even if the problem is simplified to steady solutions and the number of layers to two (meaning that each reservoir has two layers of densities ρ_1 and ρ_2 with different thicknesses), the solution is not trivial, as studies spanning almost four decades substantiate.

In the steady state, taking the measurements in a coordinate system that moves with the phase speed of any internal long wave (with infinitesimally small amplitude), the time derivative term in (2) may be dropped, and integration gives the Bernoulli expression for the velocity in the ith layer:

$$\rho_i \tfrac{1}{2}U_i^2+g \sum_{j=1}^{i} \Delta\rho_j h_{j-1} = \text{constant}. \tag{5a}$$

If $U_i = 0$ in one layer, such a stagnant layer has no proper streamlines and (5a) reduces to yield a hydrostatic pressure relationship. Any line in a stagnant layer – including its interfaces with the adjacent active layers – is then a potential streamline and equation (5a) is still valid along those. Expanding the $\Delta\rho_j$ into $(\rho_j-\rho_{j-1})$ as defined above, an alternative formulation is

$$\rho_i \tfrac{1}{2}U_i^2-g\rho_0 h_0+g \sum_{j=1}^{i-1} \rho_j H_j+g\rho_i h_{i-1} = \text{constant}. \tag{5b}$$

In (5b) the summation term denotes the additional hydrostatic pressure of the $i-1$ layers on top of layer i, which together with the term $g\rho_0 h_0$ equals the total hydrostatic pressure, P_i, on top of the ith layer. The integration constants can be determined by the fact that in the upstream reservoir the velocity U_i is negligibly small and the H_j or h_j

are known. A common procedure (Killworth 1992) is to introduce a Bernoulli PE-function B_i, representing the potential energy for each layer where P_i is the pressure on top of the ith layers:

$$B_i = P_i + g\rho_i h_{i-1}. \tag{6}$$

A constant atmospheric pressure may of course be added to this, but since it will be assumed that it does not vary over the contraction, it could be considered as shifted over to be contained in the right-hand side of (5a) and (5b). The latter equation then simplifies to be constant along the layer interfaces:

$$\rho_i \tfrac{1}{2} U_i^2 + B_i = \text{constant}. \tag{7a}$$

The flow is thus accelerated from higher B-values to lower. If the atmospheric density is approximated to be zero and the reference datum is set to coincide with the rigid lid, then the Bernoulli PE-function of the top layer is

$$B_1 = \rho_1 g\, da. \tag{7b}$$

Here da is the sea level difference relative to the datum. Since it enters the Bernoulli PE-function of all layers equally, it may be replaced by specifying the net volume flow (Armi 1986). In the present formulation, it will be retained, since doing so facilitates the calculation of the exchange flow. In pursuing the steady solution, the time-derivative term in equation (3) is set to zero:

$$U_{i,x} H_i W_i = -U_i (H_i W_i)_{,x}. \tag{8}$$

This expression for $U_{i,x}$ is inserted into equation (2). Assuming steady state, the expanded derivative on the right-hand side of equation (8) becomes

$$\rho_i U_i^2 H_i^{-1} W_i^{-1} (H_i W_{i,x} + W_i H_{i,x}) = g \sum_{j=1}^{i} \Delta\rho_j h_{j-1,x}. \tag{9}$$

For vertical sidewalls, the narrowest section for all layers coincides, and

$$W_{i,x} = 0 \tag{10}$$

for all i simultaneously. Other arrangements of sidewalls that meet this requirement could be included in this simplification, provided at least that the channel with constant centreline depth is symmetric about its mid-depth plane, which is the condition found by Dalziel (1992) for two-layer flow.

Substituting H_i from (4) into (9) yields

$$\rho_i \frac{U_i^2}{gH_i} (h_{i-1,x} - h_{i,x}) = \sum_{j=1}^{i} \Delta\rho_j h_{j-1,x}. \tag{11}$$

Introducing layer Froude numbers $F_i = [U_i^2/(gH_i)]^{1/2}$, expanding equation (11) for the $(i+1)$th layer and taking advantage of the recursive structure (i.e. that the left-hand side of (11) for layer i is identical to the left-hand side of the same equation for layer $(i-1)$ plus the term $\Delta\rho_i h_{i-1,x}$) one gets

$$\rho_{i+1} F_{i+1}^2 (h_{i,x} - h_{i+1,x}) = \rho_i F_i^2 (h_{i-1,x} - h_{i,x}) + \Delta\rho_{i+1} h_{i,x}. \tag{12a}$$

Collecting the terms for the respective layers, noting that $h_{n,x}$ is zero since the bottom is assumed horizontal (or the contraction coincides with a sill), equation (12a) renders n equations, one for each interface for the n unknowns h_k, $k = 0, 1, 2, \ldots, n-1$, with the right-hand sides equal to zero. This set of equations has the trivial solution $h_{k,x} = 0$ – meaning a non-control case – or the determinant equation must be satisfied:

$$\begin{vmatrix} \rho_1 F_1^2-\Delta\rho_1, & -\rho_1 F_1^2, & 0, & 0, & & \cdots & & \\ -\rho_1 F_1^2, & \rho_1 F_1^2+\rho_2 F_2^2-\Delta\rho_2, & -\rho_2 F_2^2, & 0, & & \cdots & & \\ 0, & -\rho_2 F_2^2, & & & & \ddots & & \\ \cdots & 0, & -\rho_i F_i^2, & \rho_i F_i^2+\rho_{i+1} F_{i+1}^2-\Delta\rho_{i+1}, & -\rho_{i+1} F_{i+1}^2, & 0, & \cdots & \\ & & & \ddots & & & & \\ 0, & \cdots & & -\rho_{n-2} F_{n-2}^2, & \rho_{n-2} F_{n-2}^2+\rho_{n-1} F_{n-1}^2-\Delta\rho_{n-1}, & -\rho_{n-1} F_{n-1}^2 & & \\ 0, & \cdots & & 0, & -\rho_{n-1} F_{n-1}^2, & \rho_{n-1} F_{n-1}^2+\rho_n F_n^2-\Delta\rho_n & & \end{vmatrix} = 0. \qquad (12b)$$

This determinant equation is equivalent to the one derived by Benton (1954), but derived more similarly to that of Baines (1988) who showed the relation between this type of derivation and the one produced by analysis of small linear perturbations of equation (2) in steady state. Only the numbering of the layers (which goes in the reverse direction, i.e. bottom up for these authors), the sign of the $\Delta\rho_j$, defined as a negative quantity by Benton (1954), and the sign of the super- and sub-diagonal elements is reversed. Changing the signs of all elements only makes the determinant change sign if the number of rows and columns, n, is odd; if n is even, the value remains the same (Aitken 1958; p. 37). As may be proved by induction, it also holds that reversing the signs of the diagonal elements in a tridiagonal determinant only changes the sign of the determinant if n is odd, otherwise not. Therefore (12b) is identical to the control criterion derived by Benton (1954) and Baines (1988).

From general theory of layered systems (e.g. Gill 1982), it is known that (12b) must have exactly n pairs of solutions, each corresponding to the velocity of a coupled long internal wave with infinitesimal amplitude displacement of the layer interfaces and going in either direction. If these solutions are real-valued, the flow is neutrally stable (Baines 1988). When approaching the constriction their passage may be counteracted by the advective flow (Benton 1954; Baines 1988). For a control to take effect in the coordinate system following the internal wave, at least one of these long internal wave speeds must be zero. Relative to the corresponding coordinate system, the velocities in the Froude numbers, F_i, in (12b) may be replaced by ones measured relative to the coordinate system of the topography and this will be considered the case from this point on.

In (12b) $\Delta\rho_1$ normally dominates over $\rho_1 F_1^2$. The physical interpretation is that the slope of the sea level, in order to balance a slope of an internal interface so that (12b) is satisfied, needs only to be an order of magnitude smaller. This is the essence of the rigid-lid approximation. If the determinant in (12b) is expanded into cofactors along the first row and the less important term is discarded, one obtains a new determinant equation where the determinant only differs from the one in (12b) in that the first row and column are omitted.

3. Decoupling of the control criterion by a stagnant layer

In order to better understand the mathematical properties of equation (12b) it is now convenient to introduce a simplification. This is done only to make the point more clearly. The following line of argument applies exactly analogously to the original more complicated determinant in (12b) that allows for arbitrary density jumps between the layers. The layers are now regarded as discrete representations of a continuously stratified liquid and one may choose to subdivide the layers so that all $\Delta\rho_j$ $(j = 2, 3, \ldots, n)$ become equal $(= \Delta\rho)$. This is analogous to introducing density coordinates (Killworth 1992). For $j = 1$, $\Delta\rho_1$ denotes the density difference between the upper layer and the atmosphere. All F_i^2 in (12*b*) divided by $\Delta\rho$ will be transformed into their densimetric counterparts:

$$f_i^2 = F_i^2/(\Delta\rho/\rho_i). \tag{13}$$

Insertion into (12*b*) together with the rigid-lid assumption gives a $(n-1)\times(n-1)$ determinant equation that specifies the control criterion for the n layers (with $n-1$ free interfaces) and is thus valid under the assumptions made of equal layer density differences, zero density on top of the upper layer and the rigid-lid approximation.

Assuming further that a stagnant layer exists at the narrowest part of the contraction, the Froude number of the stagnant layer will be zero and the determinant may be separated into the product of two smaller determinants. For clarity this can be exemplified for the case of seven layers. In order to adapt the determinant to numerical computations, the signs of all elements may be reversed which, for the above-given reason does not change the zeros of the determinant equation (Aitken 1958):

$$\begin{vmatrix} 1-f_1^2-f_2^2, & f_2^2, & 0, & 0, & 0, & 0 \\ f_2^2, & 1-f_2^2-f_3^2, & f_3^2, & 0, & 0, & 0 \\ 0, & f_3^2, & 1-f_3^2-f_4^2, & f_4^2, & 0, & 0 \\ 0, & 0, & f_4^2, & 1-f_4^2-f_5^2, & f_5^2, & 0 \\ 0, & 0, & 0, & f_5^2, & 1-f_5^2-f_6^2, & f_6^2 \\ 0, & 0, & 0, & 0, & f_6^2, & 1-f_6^2-f_7^2 \end{vmatrix} = 0. \tag{14}$$

If the fourth layer is stagnant and present at the maximum contraction, then f_4^2 is zero. Mathematically this determinant may be reduced to a product of two smaller determinants while the combined rank of the two determinants remains the same, reflecting the fact that the number of free interfaces is unchanged:

$$\underset{\text{surface layers:}}{\begin{vmatrix} 1-f_1^2-f_2^2, & f_2^2, & 0 \\ f_2^2 & 1-f_2^2-f_3^2, & f_3^2 \\ 0, & f_3^2, & 1-f_3^2 \end{vmatrix}} \cdot \underset{\text{bottom layers:}}{\begin{vmatrix} 1-f_5^2, & f_5^2, & 0 \\ f_5^2, & 1-f_5^2-f_6^2, & f_6^2, \\ 0, & f_6^2, & 1-f_6^2-f_7^2 \end{vmatrix}} = 0. \tag{15}$$

The implications are important. If the control condition is satisfied for one of these determinants, the other does not need to be satisfied, meaning that the slopes of the density interfaces become irrelevant for one group of layers provided that the control criterion is satisfied for the other group. The interpretation of this somewhat surprising property is that the existence of a stagnant layer completely decouples the group of layers above the stagnant layer from the group of layers beneath and *vice versa*. Owing to the presence of the stagnant layer, the internal wave modes are limited to either the upper or the lower group of layers. There are no internal wave modes comprising all layers. A similar decoupling argument is presented by Baines (1995, p. 167). In this case a critical flow layer plays the present role of the stagnant layer, as a separator between two isolated groups of wave mode solutions.

The property of decoupling thus represents a crucial point on which much of the treatment of the analytical solutions in the next section hinges. The presence of a stagnant intermediate layer much simplifies the numerical computations of self-similar solutions to be presented in the next section, even though the number of points for which it is possible to compute the flow will be limited by the number of layers used for resolving the stratification.

This type of decoupling of stagnant layers at a control in a contraction with vertical sidewalls has been observed in experiments both for discrete and for continuous multi-layer exchange flow (L. Armi, personal communication).

It may also be pointed out that the diagonal values adjacent to the stagnant layer are of type $(1-f_i^2)$-factors, unlike the other diagonal elements that are of the form $(1-f_i^2-f_{i+1}^2)$. This is a reflection of the appearance of the first row in (12*b*) which borders the stagnant liquid above the top layer, before the various approximations leading to (14) are applied. The slight but important differences of the two control criteria that correspond to the two determinant factors in (15) must be thoroughly observed. When a group of active layers is delimited by stagnant layers both on top and beneath, this situation is called the *intermediate* control condition case. The decoupling following from the assumption of a stagnant layer also means that it is possible that solutions exist where the trivial solution of $h_{j,x} \equiv 0$ for all j above or beneath the stagnant layer, or between them for the intermediate case. This means that all acceleration is taking place upstream of the contraction. This latter case will be called *non-control solutions* which are characterized by exactly horizontal slopes of the interfaces as they pass the control point at the minimum width section.

4. Self-similar solution techniques for decoupled groups of layers

Beginning with solving simpler cases first, the assumption will be retained that the Bernoulli PE-functions with regard to the density for the two basins are restrained such that the two curves for the surface density intersect only when there is no sea level difference between the reservoirs. When there is a net sea level difference (figure 4*a*, *b*) the intersection point will be shifted upward or downward depending on the sign of the difference. This restriction is necessary in order to differentiate from the intermediate cases when there are more than two groups of layers flowing in opposite directions.

The focus here will thus be on only two groups of layered solutions, and this case seems to be the one most represented in real estuaries. This assumption also means that the density distribution is such that a unidirectional flow will occur when there is no net sea level difference between the two basins. The top layer then stagnates and all the other layers move in the same direction driven by their baroclinic potential. If there is a small surface elevation, da, then the intersection point of the two B-profiles will be shifted downward to a new ρ-value. In a layered approximation this will occur for a set of discrete da-values that will make one particular layer stagnant at one time (figure 4*b*). The da-value that makes layer i stagnant will be denoted $da(i)$. It is also convenient to introduce the labels E (exterior), I (interior) and S (strait) for the site of the pressure profiles; these will be used as superscripts when no ambiguity is caused. For simplicity it is assumed that the rigid lid coincides with the surface interface with the atmosphere so that the surface elevation, da^E, is defined to be identically zero at all times. The corresponding elevation in the interior basin, da^I, will serve as an independent parameter, which together with the prespecified density profiles, will determine the flow in the contraction. This means that cases with a net flow through the strait are included.

The objective with this set-up is to compute the exchange through the contraction, which essentially means determining the da^S and the H_i^S from which the velocities and the volume flow may be calculated as a function of da^I. Last, the convention is that the density gradient is according to figure 4(*a*, *b*) with denser water closer to the surface in the exterior basin, but the analysis could be equally well performed for inverted

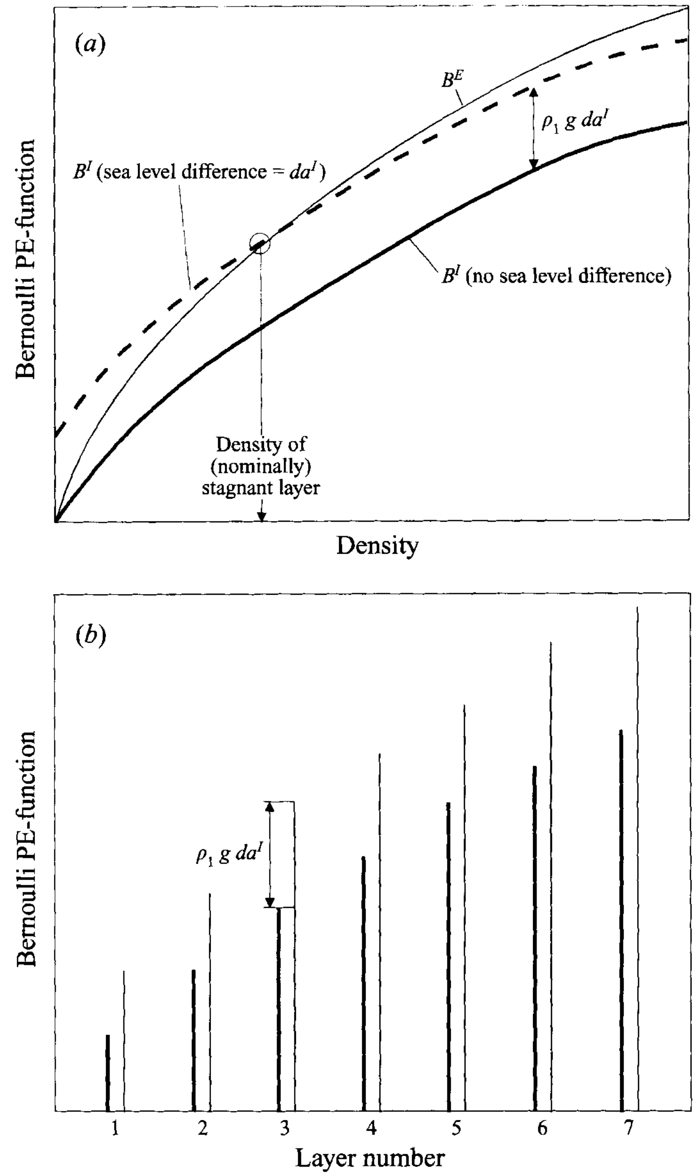

FIGURE 4. (*a*) The Bernoulli PE-function with regard to density for the interior basin (thick line) and the exterior basin (thin line). In order to restrict the solution to just two groups of active layers these only intersect for the topmost layer density. The broken line marks the vertical parallel shift of the B^I-curve by the constant pressure $\rho_1 g da^I$. The new intersection point gives the density of an infinitesimally small layer that must stagnate since it lacks Bernoulli-potential to flow. Since there may be other adjacent sections that also stagnate, this is called the nominally stagnant layer. (*b*) As (*a*) but for the discrete case where the stratification on both sides has been approximated by a subdivision into seven layers with equal density difference between adjacent layers. The increase of the Bernoulli PE-function when an additional sea level of da^I is maintained in the interior basin is marked. This is sufficient to make the third layer stagnant and is denoted $da^I(3)$.

estuaries. The chosen conventions make da^I a positive quantity. The flow direction is then $I \rightarrow E$ for the layers above the one labelled i and in the reverse direction for the bottom layers. The Bernoulli PE-function for the stagnant ith layer is then

$$B_i^X = g\rho_1 \, da^X(i) + \sum_{j=1}^{i-1} g\rho_j H_j^X - g\rho_i \sum_{j=1}^{i-1} H_j^X. \tag{16}$$

In this equation the superscript X is either I (interior) or E (external), and a sufficient condition for the ith layer to be stagnant is

$$B_i^I = B_i^E. \tag{17}$$

With the $da^E(i)$ defined to be zero, equation (17) gives the $da^I(i)$ value that makes the ith layer stagnant. Since a jump does not conserve the total energy (ordinary Bernoulli functions) for the active layers that it is composed of, it must be emphasized that in the present analysis these functions are never extended across a jump – not even across the contraction – for any active layer.

A necessary condition for these decoupled flows to exist is that the two groups of active layers leave space for the layer j to protrude into the contraction for the particular $da(j)$-value that makes it stagnant. For the fastest control mode, with layer thickness reduction 2/3,

$$(h_j^E - h_{j-1}^I) \leqslant H/2 \tag{18}$$

must hold. H is here the total depth. When the equal sign applies, the two groups of layers barely touch at the contraction; this could be called an *osculating* or 'kissing' solution (L. Armi, personal communication).

For simplicity the superscripts are temporarily dropped for the layer thickness at the sound. Let $H_0 (= da^I - da^S)$ denote the drop of sea surface to the contraction. A self-similar solution is now assumed with the height reduction factor λ, that is the ratio between the vertically contracted (at the constraint) and the uncontracted (far upstream) layer thickness. Equation (7*a*) gives the velocities of the upper layers:

$$U_i^2 = 2(\rho_1 g H_0 - \Delta\rho g(1-\lambda) \sum_{j=1}^{i-1} (i-j) H_j)/\rho_i. \tag{19}$$

A demonstration of the nature of equation (19)-type self-similar solutions, and their capacity to pass through a contraction control by satisfying the surface control criterion, will be performed for a set of n active layers. With this number of active layers, the condition that the $(n+1)$th layer is to stagnate gives that

$$\rho_1 g H_0 = \Delta\rho g(1-\lambda) \sum_{j=1}^{n} j H_{n+1-j}. \tag{20}$$

This determines the value of H_0 and, maintaining the rigid-lid assumption, the f_i^2 for the active layers ($i = 1, 2, 3, \ldots, n$) may be computed:

$$f_i^2 = 2\frac{1-\lambda}{\lambda}\left(\sum_{j=1}^{n} j H_{i+1-j} - \sum_{j=1}^{i-1} (i-j) H_j\right) \bigg/ H_i. \tag{21}$$

This set of f_i^2 is now supposed to satisfy the surface control criterion, at least for the solution found by Wood (1968) and Benjamin (1981), that is for $\lambda = 2/3$. Thus the f_i^2 of equation (21) are substituted into a surface control criterion of the type given as the first determinant of the left-hand side in (15). The numerical solutions, found by an

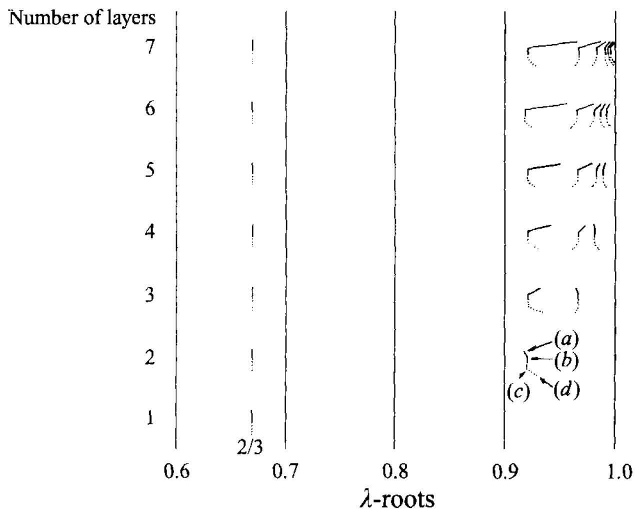

FIGURE 5. Graphical display of the λ-roots that satisfy the coupled control conditions (equation (15*b*)) both for the surface and the bottom groups of layers. Two different reservoir distributions are computed, one with unequal layer height, the other with equal layer height (i.e. linear density distribution) so that four cases (*a–d*) ensue. There are as many roots as there are layers. The corresponding roots are connected graphically from top to bottom in this order: (*a*) the roots of surface control condition, unequal layer height; (*b*) the roots of surface control condition for equal layer height; (*c*) the roots of bottom control condition for equal layer height; (*d*) the roots of bottom control condition for unequal height. The roots associated with the surface are shown solid, the others dotted. It is evident that for linear stratification the roots are close, if not identical. For unequal layer height the roots differ markedly.

interval-splitting numerical scheme, are presented in figure 5 for cases of both equal and unequal layer heights. The known solution ($\lambda = 2/3$) is reproduced, but interestingly other solutions occur as well. In fact, there are as many self-similar solutions as there are active layers. This would mean that a complete set of long internal waves associated with a critical condition at each interface is found. Each of these could be controlled at the narrowest part of the contraction depending on the downstream reservoir stratification. The corresponding height reduction factors seem to be confined to a narrow band. Except for the $\lambda = 2/3$ root, it holds approximately that $0.9 < \lambda < 1$. Note that with the assumptions made H_0 must be equal to or less than da. This is a condition that must be observed in particular when the stratification in the reservoirs becomes similar.

Analogously the same self-similar approach can be attempted for a group of n active layers beneath a stagnant one on top and above a flat bottom. Since the density of the stagnant layer now differs only by $\Delta\rho$ from the top active layer, ρ_1 in (19) now should be replaced by $\Delta\rho$. If the active layers are renumbered to run from 1 to n, the geometry gives

$$H_0 = (1-\lambda)\sum_{j=1}^{n} H_j. \tag{22}$$

Equation (21) will be modified as the rigid-lid assumption no longer holds and the displacement H_0 is, in this case, not negligible:

$$f_i^2 = 2\frac{1-\lambda}{\lambda}\left(iH_0 - \sum_{j=1}^{i-1}(i-j)\,H_j\right)\Big/ H_i. \tag{23}$$

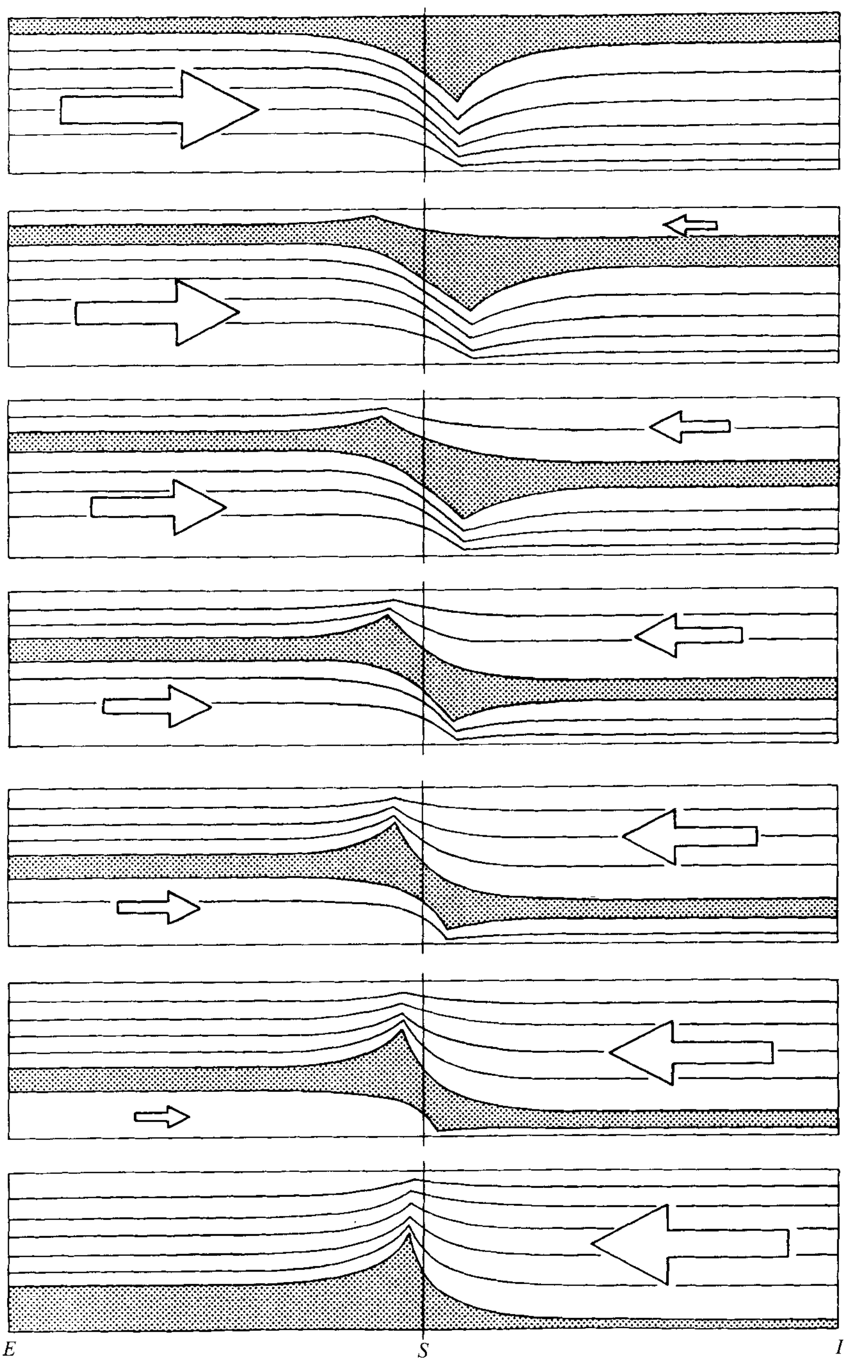

FIGURE 6. Sketches of streamlines along isopychs for a suite of decoupled solutions with seven discrete layers. The actual vertical height reduction at the strait contraction (2/3) is the essential attribute depicted. Each of the figures corresponds to a sea level elevation that makes one layer at a time stagnant.

Again, there are as many solutions as there are active layers. For the case with equal layer heights, these roots coincide within the numerical resolution with the roots found for the surface case, but for other geometrical thickness distributions there are no coinciding λ-roots except for the 2/3 solution, see figure 5. This could be expected from the seemingly symmetric arrangement with a flat top and bottom respectively in the two cases, but even if the layer order is reversed to be mirror-symmetrical, the roots that satisfy the surface and the bottom control criteria will still be slightly, but yet significantly, deviating. A suite of exact decoupling solutions is presented in figure 6.

As in the case with an intermediate group of layers surrounded by a stagnant layer both on top and below, the local Froude numbers may be computed showing the same properties with regard to the height reduction factor as above:

$$f_i^2 = 2\frac{1-\lambda}{\lambda}\left(\frac{i}{n+1}\sum_{j=1}^{n} jH_{n+1-j} - \sum_{j=1}^{i-1} jH_{i-j}\right)\Big/ H_i. \tag{24}$$

In fact, the rigid-lid assumption could be relaxed. If the atmosphere had, instead of the assumed zero density, a density that is $\Delta\rho$ less than the density of the top layer, (24) would be the adequate equation for the surface layers. The assumptions of negligible atmosphere density, Boussinesqian density distribution (equation (1)) and a rigid lid are thus not independent. From any two of these assumptions, the third follows.

5. Discussion

The volume transport through the strait that is sought may thus be completely determined by the shallow water equations and be in accordance with the control criterion and in harmony with the downstream conditions for the discrete number of sea level differences that fulfil the condition given in (18). In an application one distinct advantage is that the solutions found are exact and there is no dependence on any (semi-) empirical parameters. The exchanged density-determining properties (salinity and temperature) that in timescales longer than the assumed quasi-stationary scale will change the density profile of the basins, may be computed as the response of the basins to the exchange (e.g. Stigebrandt 1990; p. 609). The disadvantage is that the computation is only possible for a limited number of sea level differences or equivalently for a set of net barotropic volume flows, maximally as many as there are layers. There are two ways to expand the applicability.

The first is self-evident. For layered approximations of continuous profiles one is free to choose the number of layers. A coarse subdivision gives faster computation at the expense of the resolution and *vice versa*. A sensible compromise is to arrange for the osculating solutions mentioned earlier.

The second method focuses on the dynamics of the shearing interface in a flow situation such that the sea level has been slightly adjusted so that a formerly stagnant layer has just begun to flow. Still addressing the case with only two groups of active layers, these could now be called 'loosely coupled'. Consider now what happens if the densities of all the layers above the upper shearing layer are replaced by this layer's density (denoted ρ^+) and simultaneously the sea level is lowered by a positive quantity da^+

$$da^+ = \sum (\rho^+ - \rho_j)\, H_j / \rho^+, \tag{25}$$

where the summation should be taken over all the layers above the shearing interface. The dynamics of the interface cannot be changed by this transformation, at least not before it reaches the possible jump after passing through the control point. The flow situation for the shearing layers remains the same. In particular this applies at the minimum-width control point. Likewise, all the layer densities below the lower shearing layer may be replaced by the same density as this layer without changing the flow regime, with the possible exception that the jump after passing the control could be affected. Between the jumps, the two-layer dynamics are approximately the same as for the multi-layer situation they have replaced. This means that the loosely coupled flow cases may be approximately computed relying on the well-established two-layer theory (Armi 1986; Armi & Farmer 1986) and the transformation is valid for the whole continuous set of sea level differences da between two consecutive exact solutions. The transitions between the exact solution and these approximate solutions are smooth, since the same transformation applied to layers above the stagnant one, and the adjusted sea level will be lowered to become exactly horizontal on top of a homogeneous layer of the stagnant layer's density.

Such a two-layer approximation may be extended to fully coupled flow cases that do not fulfil (18). One complication is then, however, that more than one layer may become stagnant, increasing the density jump by a multiple of $\Delta\rho$ over the shearing layers. This not only means that more than one two-layer approximation may have to be computed before finding the corresponding multi-layer self-similar solution with a shearing interface height at the control that matches the stagnation condition requirement on the Bernoulli PE-functions. It also means that progressively, as the density distribution in the two reservoirs becomes closer to a lock-exchange two-layer situation, the more intense the production of turbulent mixing by instabilities will be with the increased shear (Baines 1995).

Benjamin (1981) proves that unidirectional self-similar flow through a contraction with vertical sidewalls maximizes the transfer of momentum or 'flow-force'. From this it follows directly that the decoupled exact solutions found also do so. It is not known by the present author if the coupled flow cases have this property, nor if it applies to other than vertical sidewall arrangements. In the latter case, the layers in a group will in any case not reduce their thickness as if the group consisted of a homogeneous layer (Wood 1968). Even if it could be proven to be a true theorem for the coupled cases, it would not be an attractive basis for numerical computation. The reason is that even though the momentum transfer is readily computed and the parameter space for multi-layer self-similar flow consists of only two degrees of freedom (i.e. the height of shearing interface and the da^S) the possible blocking of one or more layers makes it computationally inefficient to evaluate the loci in the parameter space that are compatible with the uncoupled control condition and the requirement that the flow cannot go against Bernoulli PE-function gradients upstream of the jumps.

If there is a sill that coincides with the maximum contraction, it also seems possible to find exact decoupled solutions. An example from the strait Oxdjupet in the Stockholm archipelago is given in figure 7. In addition to the sea level condition to make one layer stagnant at a time, the lower group of active layers must also be bounded by stagnant layers that are blocked off by the sill. In spite of the basic similarity with the non-sill case, there are sufficient complications to motivate that the sill case merits a separate treatment (Armi & Farmer 1986; Farmer & Armi 1986).

One may ask if the solutions with λ-roots greater than $2/3$ will occur in real flow situations. These roots correspond to wave modes considerably slower than the fastest one. There is certainly no mathematical reason to exclude them and laboratory experiments also indicate that they exist (L. Armi, personal communication). They appear, however, in flow situations with weak driving potential (similar stratification in both reservoirs) which are such slowly flowing active layers that they are difficult to discriminate from the stagnant adjacent layers.

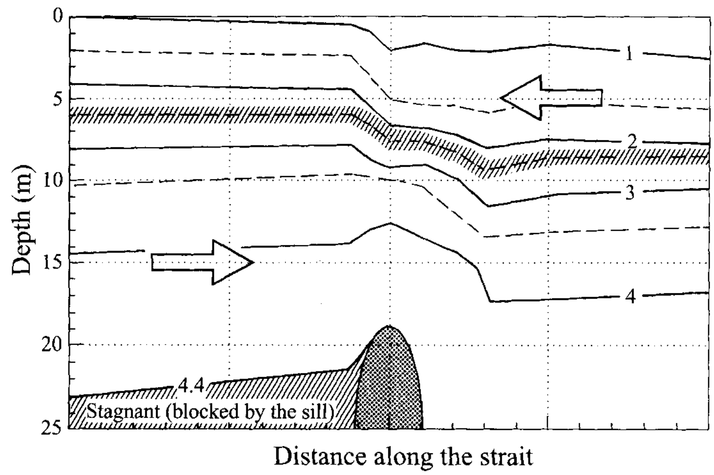

FIGURE 7. Isopychs (σ_T) observed July 13, 1995, at the strait Oxdjupet in the Stockholm archipelago. The maximum depth is about 19 m. The most distant observations from the sill are approximately one nautical mile to either side. Allowing one metre for a stagnant layer (shown cross-hatched around the 2.5-isopych), the height reduction of the upper layers is 0.76 and for the lower layers 0.71. The theoretically expected 2/3-ratio would probably be attained more closely if the most distant measurements were even farther apart. This indicates thus that decoupled solutions may be extended to silled straits.

As has been touched upon above, it is certainly possible to loosen the condition that the Bernoulli PE-functions of the two basins intersect at only one point, permitting a series of groups flowing in alternate and opposing directions as long as the stagnant layers protrude to the control point and all the groups stay decoupled. The surface elevation in the contraction would still be possible to deduce according to (20) and the suggested procedure could be pursued from the top toward the bottom. The coupled cases become more complicated, however. The two-layer approximation gives a partial explanation for why two groups of contraflowing layers behave as if they were homogeneous (Wood 1968; Benjamin 1981) provided that the sidewalls are vertical. Pursuing the same approach for several groups leads into recursive adaption numerical schemes. If one succeeded in solving these, the multi-layer problem would be reduced to instances of few-layer problems for which only a limited number of exact solutions are known, for example three layers (Long 1977). This is the basic rationale for restraining the stratification to Bernoulli PE-functions that intersect only at one point. At present only the exact decoupled cases are analytically tractable, with a possible extension to loosely coupled cases where the two-layer approximation gives an approximate numerical solution. This provides only a limited set of solutions, however. The corresponding solutions to fill the gaps must wait until the theory for more than two layers is better developed. A general three-layer solution is still in high demand (Pratt 1990).

6. Summary

For bidirectional quasi-stationary exchange through a contraction without a sill between two reservoirs with prespecified density profiles, self-similar solutions to the shallow water equations that are compatible with the control criterion (Benton 1954; Baines 1988) are computed. This has been achieved by first subdividing the assumed continuous density distribution into a numerically tractable number of assumed homogeneous layers and then separating the full bidirectional solution into two groups of opposing layers that are completely decoupled if physically vertically bounded by a stagnant layer that protrudes into the control section. If the stagnant layer is depleted there, so that a non-stagnant case occurs, this situation may be approximately solved numerically by a transformation into an equivalent two-layer flow regime.

The control section that controls the dominant fastest propagating wave mode, determined by the density distribution of the two reservoirs, coincides with the narrowest part of the connecting sound between the two basins. The entire exchange flow may be calculated for discrete sea level differences that make each of the layers stagnant one at a time. This means that from an application point of view, the functional relation may be computed between the barotropic and the baroclinic component for a suite of exchange flows that is driven by the gravitational potential of the prespecified density distribution in the reservoirs. This presupposes that the Bernoulli PE-function profile in one basin is strictly smaller than that in the other when there is no sea level difference between the basins. This constraint limits the exchange to only two groups of layers going in opposite directions. In combination with a functional description response of the basins to the exchanged properties, this serves as a basis for an exchange model free from empirical parameters. For both the upper and lower groups of layers there appear as many height reduction factor (λ) solutions as there are active layers. These are found in a range between 2/3 and 1. The fastest wave mode corresponds to a height reduction factor 2/3. If this mode is suppressed by the downstream stratification condition, the second fastest mode will be the one controlled. If this one in turn is drowned by an incident wave from the downstream side, the third takes over and so forth. A non-control regime occurs when the Bernoulli PE-functions of the two reservoirs are so similar that the flow cannot reach the downstream reservoir for any of reduction factors. In such a non-control case the interfaces are completely horizontal when passing the maximum contraction.

I thank Anders Stigebrandt for suggesting this problem to me and for many inspiring discussions. He also has contributed constructive criticism to the improvement of the manuscript as have Laurence Armi and three anonymous reviewers. Leif Lundgren has drawn the figures and Ida Engqvist has assisted with language correction. This project has been financially supported by the Swedish Environmental Protection Agency.

REFERENCES

AITKEN, A. C. 1958 *Determinants and Matrices.* Oliver and Boyd.

ARMI, L. 1986 The hydraulics of two flowing layers with different densities. *J. Fluid Mech.* **163**, 27–58.

ARMI, L. & FARMER, D. M. 1986 Maximal two-layer exchange through a contraction with barotropic net flow. *J. Fluid Mech.* **164**, 27–51.

ARMI, L. & WILLIAMS, R. 1993 The hydraulics of a stratified fluid flowing through a contraction. *J. Fluid Mech.* **251**, 355–375.

BAINES, P. G. 1988 A general method for determining upstream effects in stratified flow of finite depth over long two-dimensional obstacles. *J. Fluid Mech.* **188**, 1–22.

BAINES, P. G. 1995 *Topographic Effects in Stratified Flows.* Cambridge University Press.

BENJAMIN, T. B. 1981 Steady flows drawn from a stably stratified reservoir. *J. Fluid Mech.* **106**, 245–260.

BENTON, G. S. 1954 The occurrence of critical flow and hydraulic jumps in a multi-layered fluid system. *J. Met.* **11**, 139–150.

BINNIE, A. M. 1972 Research note: Hugoniot's method applied to stratified flow through a constriction. *J. Mech. Engng Sci.* **14**, 72–73.

CRAYA, A. 1951 Critical regimes of flows with density stratification. *Tellus* **3**, 28–42.

DALZIEL, S. B. 1991 Two-layer hydraulics: a functional approach. *J. Fluid Mech.* **223**, 135–163.

DALZIEL, S. B. 1992 Maximal exchange in channels with nonrectangular cross sections. *J. Phys. Oceanogr.* **22**, 1188–1206.

FARMER, D. M. & ARMI, L. 1986 Maximal two-layer exchange over a sill and through the combination of a sill and contraction with barotropic flow. *J. Fluid Mech.* **164**, 53–76.

GILL, A. E. 1977 The hydraulics of rotating-channel flow. *J. Fluid Mech.* **80**, 641–671.

GILL, A. E. 1982 *Atmosphere–Ocean Dynamics*. Academic.

KILLWORTH, P. D. 1992 On hydraulic control in a stratified fluid. *J. Fluid Mech.* **237**, 605–626.

LEE, J. D. & SU, C. H. 1977 A numerical method for stratified shear flows over a long obstacle. *J. Geophys. Res.* **82**, 420–426.

LONG, R. R. 1953 Some aspects of the flow of stratified fluids. I. Theoretical investigation. *Tellus* **5**, 42–58.

LONG, R. R. 1954 Some aspects of the flow of stratified fluids. II. Experiments with two-fluid system. *Tellus* **6**, 97–115.

LONG, R. R. 1955 Some aspects of the flow of stratified fluids. III. Continuous density gradients. *Tellus* **7**, 341–357.

LONG, R. R. 1977 Three-layer circulation in estuaries and harbors. *Tellus* **7**, 415–421.

MCCLIMANS, T. A. 1990 Role of laboratory experiments and models in the study of sea strait processes. In *The Physical Oceanography of Sea Straits* (ed. L. J. Pratt), pp. 373–388. Kluwer.

PRATT, L. J. ed. 1990 *The Physical Oceanography of Sea Straits*. Kluwer.

STIGEBRANDT, A. 1990 On the response of the horizontal mean vertical density distribution in a fjord to low-frequency density fluctuations in the coastal water. *Tellus* **42A**, 605–614.

STOMMEL, H. & FARMER, H. G. 1952 Abrupt change in width in two-layer open channel flow. *J. Mar. Res.* **11**, 205–214.

STOMMEL, H. & FARMER, H. G. 1953 Control of salinity in an estuary by a transition. *J. Mar. Res.* **12**, 13–20.

SU, C. H. 1976 Hydraulic jumps in an incompressible stratified fluid. *J. Fluid Mech.* **73**, 33–47.

WOOD, I. R. 1968 Selective withdrawal from a stably stratified fluid. *J. Fluid Mech.* **32**, 209–223.

WOOD, I. R. 1970 A lock exchange flow. *J. Fluid Mech.* **42**, 671–687.